CALCULATION OF NONSTATIONARY FINITE-AMPLITUDE WIND WAVES

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Reviews of the present state of wind-wave investigation can be found in many papers (for instance, [1, 2]). A method for solving the problem of motion of finite-amplitude internal waves was proposed in [3]. However, the algorithm used there did not prove useful in the case of wind waves. In connection with this, we propose here a new algorithm. The calculation results are in agreement with data obtained in observing actual wind waves.

§ 1. Plane motion of two incompressible, nonviscous, and nonmixing fluids with different densities in a gravitational field is contemplated. The flow is assumed to be continuous everywhere in the plane, potential at points away from the boundary line separating the fluids, and periodic in the horizontal direction.

Assume that the axis of ordinates of the x, y Cartesian coordinate system is directed vertically upward (Fig. 1). In the upper D_1 and the lower D_2 flow regions, the fluid velocity $\mathbf{V} = (V_x, V_y)$ satisfies the equations

$$\operatorname{div} \mathbf{V} = 0, \ \operatorname{rot} \mathbf{V} = 0, \ (x, y) \notin L, \tag{1.1}$$

the periodicity condition

$$\mathbf{V}(x + \lambda, y, t) = \mathbf{V}(x, y, t), \ (x, y) \notin L \tag{1.2}$$

and the following boundary conditions: Flow-velocity perturbations are damped with increasing distance from the boundary line L,

$$\mathbf{V}(x, y, t) \to \begin{cases} (-v_\infty - u, 0), & y \to +\infty, \\ (v_\infty - u, 0), & y \to -\infty, \end{cases} \tag{1.3}$$

the fluids do not flow across the boundary line,

$$\mathbf{V}_j \cdot \boldsymbol{\nu} = \mathbf{w} \cdot \boldsymbol{\nu}, \ j = 1, 2, \ (x, y) \in L, \tag{1.4}$$

and the Laplace law holds for the hydrodynamic pressure drop across the boundary line,

$$p_1 - p_2 = \mu k, \ (x, y) \in L, \tag{1.5}$$

where λ is the wavelength; t is the time; v_∞ = const is half the wind velocity; u = const; $\boldsymbol{\nu}$ is the unit vector of the normal to L, outward for the domain D_2; w is the velocity of points of the line L; $\mathbf{V}_j$ and p_j are the limiting

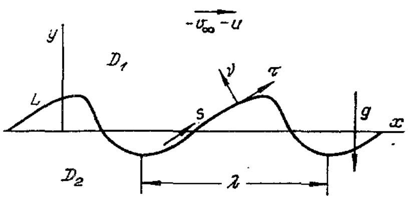

Fig. 1

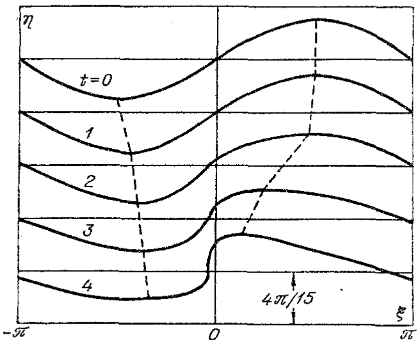

Fig. 2

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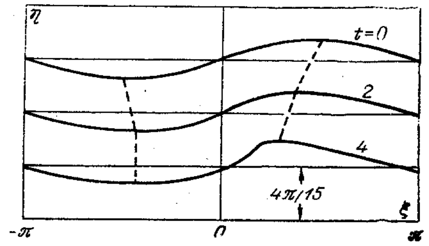

Fig. 3

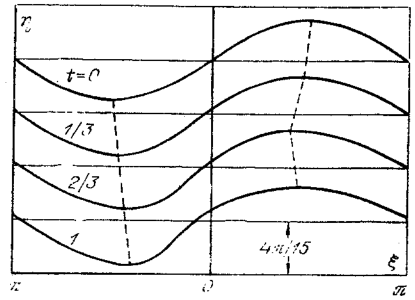

Fig. 4

values of the velocity **V** and the pressure p, respectively, in approaching L from the domain D_j, j = 1, 2; μ is the surface tension coefficient; and k is the curvature of the boundary line; k > 0 if the domain D_1 is convex in the neighborhood of the point under consideration.

It is assumed that the initial velocity field

$$\mathbf{V}(x, y, 0) = \mathbf{V}_0(x, y) \tag{1.6}$$

is known and that it satisfies conditions (1.1)-(1.4).

The problem (1.1)-(1.6) is nonlinear, since condition (1.5) is nonlinear with respect to the flow velocity field, while the boundary line L and the velocity with which it moves $\mathbf{w}\cdot\boldsymbol{\nu}$ are unknown for t > 0.

§ 2. Assume that the curve L is smooth and that it does not have singular points. We introduce the quantity $\mathbf{v} = (v_x, v_y)$ by means of the equation

$$v_x - iv_y = -u + \frac{1}{2\lambda i}\int_0^{l(t)} \gamma(\sigma, t)\,\mathrm{ctg}\,\frac{\pi}{\lambda}[\zeta(s, t) - \zeta(\sigma, t)]\,d\sigma,$$

where l is the length of the wave contour; s and σ are the arc abscissas; the positive direction of movement along the contour L is that for which the domain D_1 stays on the left (see Fig. 1); $\gamma(s, t) = (\mathbf{V}_2 - \mathbf{V}_1)\cdot\boldsymbol{\tau}$ is the intensity of the vortex sheet; $\boldsymbol{\tau}$ is the unit vector of the tangent to L, oriented in the direction of increase in the arc abscissa; and $\zeta(s, t) = \xi + i\eta$ is the complex coordinate of points of the boundary line. It is assumed that the functions γ and $\partial\zeta/\partial s$, as functions of the arc abscissa, are continuous according to Hölder, while the integral has the principal meaning of Cauchy.

In deriving the equations of motion of an internal wave, in [3] the tangential component $w_\tau = \mathbf{w}\cdot\boldsymbol{\tau}$ of the velocity of its points was assumed to be equal to $w_\tau = \mathbf{v}\cdot\boldsymbol{\tau}$. In this case, the equations have the simplest form. However, the algorithm for the numerical solution of these equations composed in [3] proved to be inadequate for calculating the motion of wind waves (where $v_\infty \neq 0$). This is due to the fact that, in the course of time, the calculation points tend to concentrate in one section of the wind wave and thin out in another. Attempts to introduce in the algorithm a uniform redistribution of calculation points along the wave contour did not improve the algorithm, since this procedure impaired considerably its stability.

This trend in the behavior of the calculation points can be eliminated by putting $w_\tau \equiv 0$. In this case, the equations of wave motion are derived as in [3]. In this, we pass from the Eulerian arc abscissa $s \in [0, l(t)]$ to the Lagrangian variable $a \in [-\pi, \pi]$ with a time correspondence between the points of the wave profile given by

$$\left(\frac{\partial\xi}{\partial t}(a, t), \frac{\partial\eta}{\partial t}(a, t)\right) = (\mathbf{v}\cdot\boldsymbol{\nu})\,\boldsymbol{\nu};$$

we introduce the function

$$\Gamma(a, t) = \gamma(s(a, t), t)|\zeta_a(a, t)|;$$

it is assumed that, at a time close to t = 0, the derivatives $\partial\Gamma/\partial a$ and $\partial^3\zeta/\partial a^3$ exist and are continuous according to Hölder as functions of a Lagrangian variable. Then the equations of wave motion with respect to the functions $\Gamma(a, t)$ and $\zeta(a, t)$ are given by

$$\bar{\zeta}_t(a, t) = \frac{i}{\zeta_a(a, t)}\,\mathrm{Im}\,\{\zeta_a(a, t)\,\bar{v}(a, t)\}; \tag{2.1}$$

$$\Gamma_t(a, t) + R\int_{-\pi}^{\pi} \Gamma_t(\alpha, t)\,K(a, \alpha, t)\,d\alpha = H(a, t), \tag{2.2}$$

where

$$\bar{v}(a, t) = -u + \frac{1}{2\lambda i}\int_{-\pi}^{\pi}\Gamma(\alpha, t)\operatorname{ctg}\frac{\pi}{\lambda}[\zeta(a, t) - \zeta(\alpha, t)]\,d\alpha;$$

$$K(a, \alpha, t) = \frac{1}{\lambda}\operatorname{Im}\left\{\zeta_a(a, t)\operatorname{ctg}\frac{\pi}{\lambda}[\zeta(a, t) - \zeta(\alpha, t)]\right\};$$

$$H(a, t) = 2R\operatorname{Re}\left\{\zeta_a\frac{\pi}{\lambda^2 i}\int_{-\pi}^{\pi}\Gamma(\alpha)\frac{\zeta_t(a) - \zeta_t(\alpha)}{1 - \cos\frac{2\pi}{\lambda}[\zeta(a) - \zeta(\alpha)]}\,d\alpha\right\} +$$

$$+ \frac{\partial}{\partial a}\left\{\frac{2\mu k}{\rho_1 + \rho_2} - \gamma v_\tau - R\left(\frac{\gamma^2}{4} + 2g\eta + v_\tau^2 - v_\nu^2\right)\right\} - 2R\operatorname{Re}\{\zeta_{ta}\bar{v}\},$$

where ρ_j is the density of the fluid in the domain D_j, $j = 1, 2$; the parameter $R = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$; g is the acceleration due to gravity; and $v_\tau - iv_\nu = \bar{v}\zeta_a/|\zeta_a|$.

The initial problem (1.1)-(1.6) is equivalent to the Cauchy problem for the system of integrodifferential equations (2.1), (2.2) with the initial data

$$\Gamma(a, 0) = \Gamma_0(a),\ \zeta(a, 0) = \zeta_0(a), \tag{2.3}$$

where the functions Γ_0 and ζ_0 are assigned. The function Γ_0 must satisfy the condition

$$\int_{-\pi}^{\pi}\Gamma_0(a)\,da = 2v_\infty\lambda.$$

It should be noted that system (2.1), (2.2) differs considerably from the similar system in [3].

The numerical solution of problem (2.1)-(2.3) is obtained as in [3] by using the Taylor formula,

$$\Gamma(a, t + \Delta t) = \Gamma(a, t) + \Gamma_t(a, t)\Delta t + \Gamma_{tt}(a, t)(\Delta t)^2/2; \tag{2.4}$$

$$\zeta(a, t + \Delta t) = \zeta(a, t) + \zeta_t(a, t)\Delta t + \zeta_{tt}(a, t)(\Delta t)^2/2 + \zeta_{ttt}(a, t)(\Delta t)^3/6. \tag{2.5}$$

The value of the interval Δt ensuring the stability of calculations is chosen by performing trial calculations with different intervals Δt until the interval is smaller than a certain critical value Δt_*. The following invariants of system (2.1), (2.2) are used for checking the calculation accuracy:

$$\int_{-\pi}^{\pi}\Gamma(a, t)\,da = 2v_\infty\lambda,\quad \int_{-\pi}^{\pi}\bar{v}(a, t)\,\zeta_a(a, t)\,da = -u\lambda.$$

In order to suppress short-wave instability [3], smoothing-out is used for each interval of the calculated values of the functions Γ, ζ_t, Γ_t, ζ_{tt}, Γ_{tt}, and ζ_{ttt}. In this, the smoothed-out value of the function at a certain point is understood as the value at this point of a third-power polynomial which approximates the function with respect to its values at the given point and six neighboring points (three on the left and three on the right) according to the method of least squares.

The program realizing this algorithm has been composed in the AL'FA-6 language for the BÉSM-6 computer. For 60 calculation points on the wave contour, the calculation of a single step requires 16 sec of computer time in computation based on Eqs. (2.4) and (2.5), and 11 sec in computation without an allowance for the derivatives Γ_{tt} and ζ_{ttt}.

§ 3. We shall provide examples of calculations of wind waves at the water–air interface ($R = 0.9975$). In the cases considered below, $\lambda/(2\pi)$ and $\lambda/(2\pi v_\infty)$ are used as the units of length and time. The dimensionless parameters $Fr = g\lambda/(2\pi v_\infty^2)$ and $W = \mu/(\rho_1 + \rho_2)\{\lambda/(2\pi)\}^{-1}v_\infty^{-2}$ are the Froude and Weber numbers. The constant u is assigned so that the x, y coordinate system moves at the velocity of a wave with an infinitesimal amplitude. The initial velocity field is chosen in correspondence with the linear theory [4].

Gravitational Wind Wave. Variant 1: $FR = 0.2556$; $u = 0.4975$; $\zeta(a, 0) = a + i0.2\pi \sin a$; $\Gamma(a, 0) = 2 + 0.199\pi \cdot \sin a$. Variant 2: $Fr = 0.64$; $u = 0.2$; $\zeta(a, 0) = a + i0.1\pi \sin a$; $\Gamma(a, 0) = 2 + 0.04\pi \sin a$. The calculations are performed for the time up to $t = 4$ with intervals $\Delta t = 1/10$ in both variants. The wave shapes for different instants of time are given in Figs. 2 and 3 (variants 1 and 2, respectively), where the wave peaks and troughs are connected by dashed straight-line segments. The wave evolution is characterized by the following features: The wave symmetry is disturbed; the windward nodes move at the velocity of infinitesimal-amplitude waves; the leeward slopes of waves become steeper, while the windward slopes become flatter. The latter trend is more strongly pronounced in the variant with steeper waves. It should be mentioned that, in the first variant (in contrast to the second variant), the steepness and velocity of a wave affecting the stability of its evolution lie outside the wave slope vs wave velocity diagram which holds for actual waves ([1], Fig. 6.4-2). The above peculiarities in the evolution of wind waves are in agreement with the results of observations in nature.

For values in the range $0 < \mathrm{Fr} < 1/R - R$, a trend opposite to that described above prevails: The leeward wave slopes become flatter, while the windward slopes become steeper.

Capillary Wind Wave. Variant 3: $W = 3.995$; $u = -1$; $\zeta(a, 0) = a + i0.2\pi \sin a$; $\Gamma(a, 0) = 2 - 0.4\pi \sin a$. The calculations are performed for the time up to the moment $t = 1$ for the interval $\Delta t = 1/90$ without taking into account the derivatives Γ_{tt} and ζ_{ttt}. The wave evolution shown in Fig. 4 (the wave peaks and troughs are also connected by dashed straight-line segments) is similar to the evolution of gravitational waves. However, it is also characterized by the fact that the wave tops become flatter and the troughs deeper.

With a reduction in the Froude and Weber numbers in comparison with those indicated in variants 1-3, the wave evolution is retarded, while the critical interval Δt_* remains almost unchanged. The latter leads to the fact that, in calculating ripple waves, the role of the nonlinear effects caused by the finiteness of the wave amplitude is not revealed even if a large amount of computer time is used. This means that the linear theory adequately describes the motion of finite-amplitude ripple waves.

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HYDRODYNAMIC STABILITY OF TWO-DIMENSIONAL POISEUILLE FLOW OF A NON-NEWTONIAN LIQUID WITH A HIGH-VISCOSITY CORE IN A COOLED CHANNEL

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Viscoplastic liquids occupy an important position among non-Newtonian liquids [1, 2]. The hydrodynamic stability of the two-dimensional Poiseuille flow of these liquids was investigated in [3, 4]. The mechanical characteristics of viscoplastic media are determined by the dimensionless rheological equation, which relates the stress tensor deviator σ_{ij} to the strain rate tensor f_{ij} [1]:

$$\begin{aligned} \sigma_{ij} &= 2\left(1 + \frac{\varkappa}{\sqrt{2f_{ij}f_{ij}}}\right) f_{ij} \quad \text{for} \quad \sqrt{\tfrac{1}{2}\sigma_{ij}\sigma_{ij}} \geqslant \varkappa, \\ f_{ij} &= 0 \quad \text{for} \quad \sqrt{\tfrac{1}{2}\sigma_{ij}\sigma_{ij}} \leqslant \varkappa, \end{aligned} \tag{1}$$

where $\varkappa = \tau_0 L/\mu U$ is the plasticity parameter; μ is the plastic dynamic viscosity; τ_0 is the ultimate shearing stress; L is the characteristic dimension (half-width of the channel); and U is the characteristic velocity. Due to the existence of the ultimate shearing stress τ_0 for a viscoplastic liquid, zones where the medium moves as a quasisolid body as well as viscous flow zones can form in the flow of such a liquid through channels [2].

The dimensionless shearing stress τ as a function of the dimensionless shearing rate δ for unidimensional shear flow of a viscoplastic liquid (1) is shown in Fig. 1. The rheological equation (1) is approximate for many actual liquids and the flow curve is essentially nonlinear for low shearing rates [5] (dashed curve in Fig. 1). The rheological law is in this case written conveniently as

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